Write your name here

Surname	Other names

Pearson Edexcel Level 3 GCE

Centre Number | Candidate Number

Further Mathematics

Advanced

Paper 1: Core Pure Mathematics 1

Sample Assessment Material for first teaching September 2017

Time: 1 hour 30 minutes

Paper Reference **9FM0/01**

You must have:
Mathematical Formulae and Statistical Tables, calculator

Total Marks

Candidates may use any calculator permitted by Pearson regulations. Calculators must not have the facility for algebraic manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions

- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- **Fill in the boxes** at the top of this page with your name, centre number and candidate number.
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided
  – *there may be more space than you need.*
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Answers should be given to three significant figures unless otherwise stated.

Information

- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- There are 9 questions in this question paper. The total mark for this paper is 75.
- The marks for **each** question are shown in brackets
  – *use this as a guide as to how much time to spend on each question.*

Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.

Turn over ▶

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Answer ALL questions. Write your answers in the spaces provided.

1. Prove that

$$\sum_{r=1}^{n} \frac{1}{(r+1)(r+3)} = \frac{n(an+b)}{12(n+2)(n+3)}$$

where a and b are constants to be found.

(5)

METHOD 1: method of differences

notice from the sigma notation that we're asked to evaluate a summation - but the fraction with the 1 in the numerator means we can't use any of our standard summations results from CP1 ∴ only left to use method of differences from CP2

↳ need to get expression for u_r in the question to look like

$\sum_{r=1}^{n} f(r) - f(r+1)$ - so looking to use PARTIAL FRACTIONS

$$u_r = \frac{1}{(r+1)(r+3)} = \frac{A}{r+1} + \frac{B}{r+3}$$

$$\Rightarrow 1 = A(r+3) + B(r+1)$$

WAY 1: by comparing coefficients

...r terms: $0 = A + B$ - ①

...constant terms: $1 = 3A + B$ - ②

solve simultaneously - calc equation solver OR by elimination

② - ①
$$\begin{array}{r} 3A + B = 1 \\ -\quad A + B = 0 \\ \hline 2A = 1 \end{array}$$

$\div 2 \qquad \div 2$

$\boxed{A = 1/2}$

sub into ①

$0 = \frac{1}{2} + B$

$\Rightarrow \boxed{B = -1/2}$

WAY 2: by substitution

let $r = -3$,

$1 = -2B$

$\div -2 \qquad \div -2$

$\Rightarrow \boxed{B = -1/2}$

let $r = -1$,

$1 = 2A$

$\div 2 \qquad \div 2$

$\Rightarrow \boxed{A = 1/2}$

$$\Rightarrow u_r = \frac{1}{(r+1)(r+3)} = \frac{1}{2(r+1)} - \frac{1}{2(r+3)}$$

$$\therefore \sum_{r=1}^{n} \frac{1}{2(r+1)} - \frac{1}{2(r+3)}$$

– this is a subtraction, so can use method of differences

WAY 1: numerical method – sub in $r=1,2,3,\ldots,n-1,n$

$$u_1: \frac{1}{2(1+1)} - \frac{1}{2(1+3)} = \frac{1}{4} - \cancel{\frac{1}{8}}$$

$$u_2: \frac{1}{2(2+1)} - \frac{1}{2(2+3)} = \frac{1}{6} - \cancel{\frac{1}{10}}$$

$$u_3: \frac{1}{2(3+1)} - \frac{1}{2(3+3)} = \cancel{\frac{1}{8}} - \cancel{\frac{1}{12}}$$

$$\vdots$$

$$u_{n-2}: \frac{1}{2(n-2+1)} - \frac{1}{2(n-2+3)} = \cancel{\frac{1}{2n-2}} - \cancel{\frac{1}{2n+2}}$$

$$u_{n-1}: \frac{1}{2(n-1+1)} - \frac{1}{2(n-1+3)} = \cancel{\frac{1}{2n}} - \frac{1}{2(n+2)}$$

$$u_n: \frac{1}{2(n+1)} - \frac{1}{2(n+3)} = \cancel{\frac{1}{2n+2}} - \frac{1}{2(n+3)}$$

after cancelling all like terms, left with:

$$\frac{1}{4} + \frac{1}{6} - \frac{1}{2(n+2)} - \frac{1}{2(n+3)}$$

getting common denominator

$$\frac{3(n+2)(n+3) + 2(n+2)(n+3) - 6(n+3) - 6(n+2)}{12(n+2)(n+3)}$$

expand further

$$\frac{3n^2+15n+18+2n^2+10n+12-6n-18-6n-12}{12(n+2)(n+3)}$$

$$= \frac{5n^2+13n}{12(n+2)(n+3)}$$ factorise 'n' out

$$\boxed{= \frac{n(5n+13)}{12(n+2)(n+3)}}$$

WAY 2: mechanical way

let $f(r) = \frac{1}{2(r+1)}$ and $f(r+2) = \frac{1}{2(r+3)}$

evaluate $\sum_{r=1}^{n} f(r) - f(r+2)$ for $r = 1, 2, 3, \ldots, n-2, n-1, n$

$u_1 : f(1) - f(1+2) = f(1) - \cancel{f(3)}$

$u_2 : f(2) - f(2+2) = f(2) - \cancel{f(4)}$

$u_3 : f(3) - f(3+2) = \cancel{f(3)} - \cancel{f(5)}$

$\vdots \qquad\qquad \vdots$

$u_{n-2} : f(n-2) - f(n-2+2) = \cancel{f(n-2)} - \cancel{f(n)}$

$u_{n-1} : f(n-1) - f(n-1+2) = \cancel{f(n-1)} - f(n+1)$

$u_n : f(n) - f(n+2) = \cancel{f(n)} - f(n+2)$

after cancelling, left with:

$f(1) + f(2) - f(n+1) - f(n+2)$

subbing into prev. defined $f(r)$

$$\Rightarrow \frac{1}{2(1+1)} + \frac{1}{2(2+1)} - \frac{1}{2(n+1+1)} - \frac{1}{2(n+2+1)}$$

$$= \frac{1}{4} + \frac{1}{6} - \frac{1}{2(n+2)} - \frac{1}{2(n+3)}$$

and manipulating as seen in WAY 1 to get:

$$\boxed{\frac{n(5n+13)}{12(n+2)(n+3)}}$$

METHOD 2: by induction

could've treated this as a summations proof by induction - i.e proving u_r is true for all $n \in \mathbb{N}$ (by proving true for $n = 1, 2, \ldots$)

step 1: base case: prove true for $n = 1$

LHS: $\sum_{r=1}^{1} \frac{1}{(r+1)(r+3)} = \frac{1}{(1+1)(1+3)} = \frac{1}{2(4)} = \frac{1}{8}$

RHS: $\frac{1(a(1)+b)}{12(1+2)(1+3)} = \frac{a+b}{12(3)(4)} = \frac{a+b}{144}$

LHS = RHS

$$\Rightarrow \frac{1}{8} = \frac{a+b}{144}$$

cross multiply

$144 = 8(a+b)$

$\div 8 \qquad \div 8$

$a + b = 18 \quad -①$

now next inductive step : prove true for $n=2$

LHS: $\sum_{r=1}^{2} \frac{1}{(r+1)(r+3)} = \frac{1}{8} + \frac{1}{(2+1)(2+3)} = \frac{1}{8} + \frac{1}{3(5)}$

$= \frac{23}{120}$

RHS: $\frac{2(2a+b)}{12(2+2)(2+3)}$

$= \frac{4a+2b}{12(4)(5)}$

equating LHS = RHS

$$\frac{23}{120} = \frac{4a+2b}{240}$$

LHS ×2 so can equate numerators

$46 = 4a + 2b$ - ②

solve ① and ② simultaneously - calc equation solver or by elimination

② - ①

$$\begin{aligned} 2a + b &= 23 \\ -\quad a + b &= 18 \\ \hline a &= 5 \end{aligned}$$

- sub into ①

$5 + b = 18$

$\Rightarrow b = 13$

Step 2: assumption step - assume 'a' and 'b' true for $n=k$

$$\sum_{r=1}^{k} \frac{1}{(r+1)(r+3)} = \frac{k(5k+13)}{12(k+2)(k+3)}$$

AIM:

$$\frac{(k+1)(5(k+1)+13)}{12(k+3)(k+4)} = \frac{(k+1)(5k+18)}{12(k+3)(k+4)}$$

Step 3: induction step - prove true for $n=k+1$

$$\sum_{r=1}^{k+1} \frac{1}{(r+1)(r+3)} = \sum_{r=1}^{k} \frac{1}{(r+1)(r+3)} + \sum_{r=1}^{1} \frac{1}{(r+1)(r+3)}$$

$$= \frac{k(5k+13)}{12(k+2)(k+3)} + \frac{1}{(k+1+1)(k+1+3)}$$

$$= \frac{k(5k+13)}{12(k+2)(k+3)} + \frac{1}{(k+2)(k+4)}$$

getting common denominator

$$\frac{k(5k+13)(k+4) + 12(k+3)}{12(k+2)(k+3)(k+4)}$$

expand numerator

$$\frac{5k^3 + 33k^2 + 52k + 12k + 36}{12(k+2)(k+3)(k+4)}$$

- factorise numerator (calc equtn solver)

Question 1 continued

$$= \frac{(k+1)\cancel{(k+2)}(5k+18)}{12\cancel{(k+2)}(k+3)(k+4)} = \frac{(k+1)(5k+18)}{12(k+3)(k+4)} = \text{AIM } (\checkmark)$$

$\therefore$ true for $n=k+1$

step 4: conclusion step:

since true for $n=1$ and 2, if true for $n=k$ and true for $n=k+1$, then true for all $n \in \mathbb{N}$

$$\Rightarrow \boxed{\sum_{r=1}^{n} \frac{1}{(r+1)(r+3)} = \frac{n(5n+13)}{12(n+2)(n+3)}}$$

(Total for Question 1 is 5 marks)

2. Prove by induction that for all positive integers n,

$$f(n) = 2^{3n+1} + 3(5^{2n+1})$$

is divisible by 17

(6)

prove by induction – prove conjecture is true for all $n \in \mathbb{N}$ – here dealing with a DIVISIBILITY PROOF

step 1: base case : prove true for $n=1$

$$f(1) = 2^{3(1)+1} + 3(5^{2(1)+1})$$
$$= 2^4 + 3(5^3)$$
$$= 16 + 3(125)$$
$$= 16 + 375$$
$$= 391 = 17(23)$$

$\therefore$ true for $n=1$

step 2: assumption step: assume true for $n=k$

$f(k) = 2^{3k+1} + 3(5^{2k+1})$ is divisible by 17 for all $k \in \mathbb{N}$

step 3: induction step

WAY 1: indices manipulation

$$f(k+1) = 2^{3(k+1)+1} + 3(5^{2(k+1)+1})$$

split indices up to get $f(k)$

$$= 2^{3k+1} \cdot 2^3 + 3(5^{2k+1})5^2$$
$$= 8(2^{3k+1}) + 3(5^{2k+1})25$$

need to split the '25' coefficient such that factor $f(k)$ out and get an expression that is a multiple of 15

$$\underbrace{25}_{8 \quad 17}$$

$$= 8(2^{3k+1} + 3(5^{2k+1})) + 17(3(5^{2k+1}))$$
$$= 8(f(k)) + 17(3(5^{2k+1}))$$

which is divisible by 17

$\therefore$ true for $n=k+1$

WAY 2: more methodical $f(k+1)-f(k)$

$$f(k+1)-f(k) = 2^{3k+4}+3(5^{2k+3})-2^{3k+1}-3(5^{2k+1})$$

splitting up the indices to get $f(k)$

$$= 2^{3k+1}(2^3)+3(5^{2k+1})(5^2)-2^{3k+1}-3(5^{2k+1})$$

$$= 8(2^{3k+1})+25(3(5^{2k+1}))-2^{3k+1}-3(5^{2k+1})$$

collect like 'terms'

$$= 7(2^{3k+1})+24(3(5^{2k+1}))$$

$$24 \to 7 \quad 17$$

split '24' coefficient so factorise $f(k)$ AND get a multiple of 17

$$f(k+1)-f(k) = 7(2^{3k+1}+3(5^{2k+1}))+17(3(5^{2k+1}))$$

$$\Rightarrow f(k+1) = 8(2^{3k+1})+17(3(5^{2k+1}))$$

$\therefore$ true for $n=k+1$

step 4: conclusion step

since true for $n=1$, if true for $n=k$ and true for $n=k+1$, then true for all $n \in \mathbb{N}$

Question 2 continued

(Total for Question 2 is 6 marks)

3. $$f(z) = z^4 + az^3 + 6z^2 + bz + 65$$

where a and b are real constants.

Given that $z = 3 + 2\mathbf{i}$ is a root of the equation $f(z) = 0$, show the roots of $f(z) = 0$ on a single Argand diagram.

(9)

notice we're given a quartic, which according to the Fundamental Law of Algebra can have the following combination of roots:

- 4 real roots
- 2 real roots and a complex conjugate pair
- 2 complex conjugate pairs

↳ because already given a complex root, z_1, know that the second root, z_2, must be the complex conjugate of z_1

following rule that if $z = a + bi$, $z^* = a - bi$
$z = 3 + 2i$, $z^* = 3 - 2i$

METHOD 1: by inspection

forming a quadratic out of the complex conjugate pair

WAY 1: roots of polynomial equtn

general rule: $z^2 - (\alpha + \beta) + \alpha\beta$

where $z + z^* = 2z_a = 2(3) = 6$
$zz^* = z_a^2 + z_b^2 = 2^2 + 3^2$

WAY 2: factor theorem - if $(x - z)$ is a factor, $f(z) = 0$

$(z - 3 + 2i)(z - 3 - 2i) = 0$

expand

$z^2 - (3 + 2i + 3 - 2i) + (3 + 2i)(3 - 2i)$

$= z^2 - 6z + 13$

and to get the quartic - need to multiply our quadratic by another quadratic - eg. $Az^2 + Bz + C$ (NOTE: in capitals so doesn't get confused by lower case in equation

$(z^2 - 6z + 13)(Az^2 + Bz + C) = z^4 + az^3 + 6z^2 + bz + 65$

... solve BY INSPECTION - compare KNOWN coefficients

... z^4:

$\Rightarrow \boxed{A = 1}$

... constants:

$13C = 65$
$\div 13 \quad \div 13$

$\Rightarrow \boxed{C = 5}$

... z^2:

$13A - 6B + C = 6$

$\Rightarrow 13(1) - 6B + 5 = 6$

$\Rightarrow 6B = 12$
$\div 6 \quad \div 6$

$\Rightarrow \boxed{B = 2}$

$\therefore$ unknown $= (z^2 + 2z + 5)$

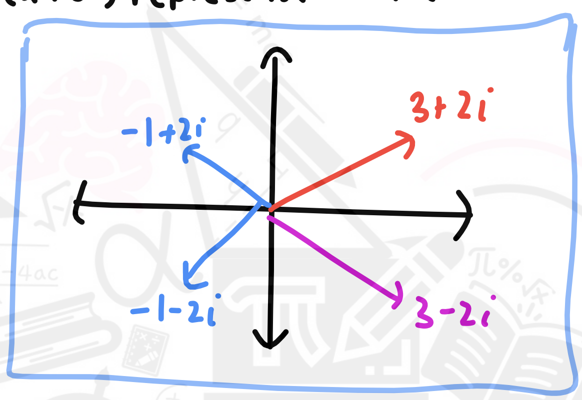
calc equtn solver or quadratic formula

$$z = \frac{-2 \pm \sqrt{(2)^2 - 4(1)(5)}}{2}$$

$$= \frac{-2 \pm \sqrt{4 - 20}}{2} = \frac{-2 \pm \sqrt{-16}}{2}$$

$$= -1 \pm 2i$$

$\Rightarrow$ 4 roots: $3 \pm 2i$, $-1 \pm 2i$ – plot on Argand diagram $\rightarrow$ complex numbers $(a+bi)$ represented with Cartesian coordinates (a,b)



METHOD 2: using factor theorem to get full quartic - then calc equation solver for roots

using fact that if $(z-\alpha)$ is a factor then $f(z) = 0$ - for given $z = 3+2i$

$f(3+2i) = 0$

$$(3+2i)^4 + a(3+2i)^3 + 6(3+2i)^2 + b(3+2i) + 65 = 0$$

evaluate on calc

$$-119 + 120i + a(-9+46i) + 6(5+12i) + 3b + 2bi + 65 = 0$$

expand brackets

$$-119 + 120i - 9a + 46ai + 30 + 72i + 3b + 2bi + 65 = 0$$

equate real and imaginary terms

...real:

$$-119 - 9a + 30 + 3b + 65 = 0$$

$$\Rightarrow 9a - 3b = -119 + 30 + 65$$

$$\Rightarrow 9a - 3b = -24 \quad -①$$

...imaginary:

$$120 + 46a + 72 + 2b = 0$$

$$\Rightarrow 46a + 2b = -192 \quad -②$$

solve simultaneously on calc equation solver

$\Rightarrow \boxed{a, b = -4}$

sub into $f(z)$

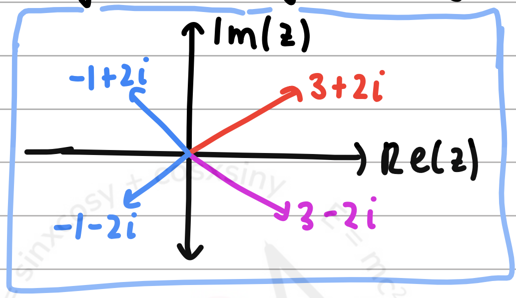
Question 3 continued

$f(z) = z^4 - 4z^3 + 6z^2 - 4z + 65$

solve for roots using calc equation solver - QUARTICS

$\Rightarrow z_1 = 3+2i,\ z_2 = 3-2i,\ z_3 = -1+2i,\ z_4 = -1-2i$

plotting on the Argand diagram



NOTE : would usually be able to do 'roots of polynomial equations' formulae but given 'b' is part of '$\Sigma\alpha\beta$' which we don't have enough information to analyse

(Total for Question 3 is 9 marks)

4.

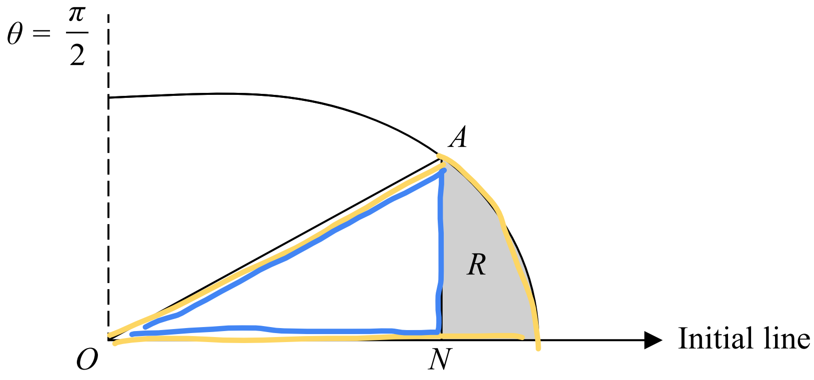


Figure 1

The curve C shown in Figure 1 has polar equation

$$r = 4 + \cos 2\theta \qquad 0 \leqslant \theta \leqslant \frac{\pi}{2}$$

At the point A on C, the value of r is $\frac{9}{2}$

The point N lies on the initial line and AN is perpendicular to the initial line.

The finite region R, shown shaded in Figure 1, is bounded by the curve C, the initial line and the line AN.

Find the exact area of the shaded region R, giving your answer in the form $p\pi + q\sqrt{3}$ where p and q are rational numbers to be found.

(9)

looking at the exact area we're asked to find - remember that with polar areas have to consider these radially

① first evaluate area of the polar segment : remember formula for integration of polar curves: $\frac{1}{2}\int_{\alpha}^{\beta} r^2 \, d\theta$

know that $\alpha = 0$ but for the β at A need to sub in given $r = 9/2$ into the polar equation to get the corresponding θ value for β

$$\frac{9}{2} = 4 + \cos 2\theta$$

$$\Rightarrow \cos 2\theta = \frac{1}{2}$$

taking $\cos^{-1}$ of both sides

$$\Rightarrow 2\theta = \cos^{-1}\left(\tfrac{1}{2}\right) = \frac{\pi}{3}, \left(\left(2\pi - \pi/3\right) = 5\pi/3 - \text{out of range}\right)$$

Question 4 continued

$\div 2 \qquad \div 2$

$\theta = \pi/6$

Sub into formula for polar integration

$$\Rightarrow \frac{1}{2}\int_0^{\pi/6} (4+\cos 2\theta)^2 \, d\theta$$

expand inside the bracket

$$= \frac{1}{2}\int_0^{\pi/6} (16+8\cos 2\theta+\cos^2 2\theta)\, d\theta$$

know can't really integrate trig powers - using cos double angle rearranged

$$\cos^2\theta = \frac{1}{2}+\frac{1}{2}\cos 2\theta$$

$$\cos^2 2\theta = \frac{1}{2}+\frac{1}{2}\cos 4\theta$$

$$= \frac{1}{2}\int_0^{\pi/6} \left(16+8\cos 2\theta+\frac{1}{2}+\frac{1}{2}\cos 4\theta\right) d\theta$$

$$= \frac{1}{2}\int_0^{\pi/6} \left(\frac{33}{2}+8\cos 2\theta+\frac{1}{2}\cos 4\theta\right) d\theta$$

integrate using $\int \cos k\theta = \frac{1}{k}\sin k\theta + c$

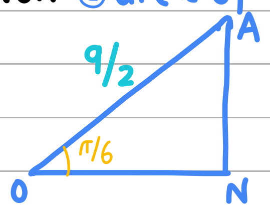
$$= \frac{1}{2}\left[\frac{33}{2}\theta + 4\sin 2\theta + \frac{1}{8}\sin 4\theta\right]_0^{\pi/6}$$

$$= \frac{1}{2}\left\{\left[\frac{33}{2}(\pi/6) + 4\sin\left(2\times\frac{\pi}{6}\right) + \frac{1}{8}\sin\left(4\times\frac{\pi}{6}\right)\right] - \left[\frac{33}{2}(0) + 4\sin(2\times 0) + \frac{1}{8}\sin(4\times 0)\right]\right\}$$

$$= \frac{1}{2}\left(\left(\frac{11}{4}\pi + 2\sqrt{3} + \frac{\sqrt{3}}{16}\right) - (0)\right)$$

$$= \boxed{\frac{11\pi}{8} + \frac{33\sqrt{3}}{32}}$$

now ② area of triangle OAN - using fact that we know that $r = 9/2$; using dfn of polar coordinate : $(r\cos\theta, r\sin\theta)$



know that $ON = x = \frac{9}{2}\cos(\pi/6) = \frac{9\sqrt{3}}{4}$

$AN = y = \frac{9}{2}\sin(\pi/6) = 9/4$

$$\Rightarrow \text{area of triangle} = \frac{1}{2}\left(\frac{9\sqrt{3}}{2}\right)\left(\frac{9}{4}\right) = \boxed{\frac{81\sqrt{3}}{32}}$$

(Total for Question 4 is 9 marks)

subbing all into overall strategy for R

$$R = \frac{11\pi}{8} + \frac{33\sqrt{2}}{32} - \frac{81\sqrt{3}}{2}$$

$$= \frac{11\pi}{8} - \frac{48\sqrt{3}}{32}$$

simplify

$$\boxed{R = \frac{11}{8}\pi - \frac{3\sqrt{3}}{2}}$$

where $p = \frac{11}{8}$, $q = -3/2$

5. A pond initially contains 1000 litres of unpolluted water.

The pond is leaking at a constant rate of 20 litres per day.

It is suspected that contaminated water flows into the pond at a constant rate of 25 litres per day and that the contaminated water contains 2 grams of pollutant in every litre of water.

It is assumed that the pollutant instantly dissolves throughout the pond upon entry.

Given that there are x grams of the pollutant in the pond after t days,

(a) show that the situation can be modelled by the differential equation,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 50 - \frac{4x}{200 + t}$$

(4)

(b) Hence find the number of grams of pollutant in the pond after 8 days.

(5)

(c) Explain how the model could be refined.

(1)

(a) noticing this is a typical 'filling the container' 1ODE question, so using the following format:

· vol. of liquid after 't' days: $1000 + (25t - 20t) = 1000 + 5t$

· concentration of pollutant after 't' days: $x / 1000 + 5t$

· rate of pollutant in : $2g \times 25 = 50g$ per day

· rate of pollutant out : $\frac{x}{1000+5t} \times 20 = \frac{20x}{1000+5t} = \frac{4x}{200+t}$

∴ by definition of $\frac{dx}{dt}$ = rate in − rate out

$$\boxed{\Rightarrow \frac{dx}{dt} = 50 - \frac{4x}{200+t}}$$

(b) now asked to solve the above 1ODE − getting in form $\frac{dx}{dt} + Py = Q$

$$\frac{dx}{dt} + \frac{4x}{200+t} = 50$$

↳ see straight away can't use solving by separation of variables as it involves the sum rather than the product of two expressions

↳ seeing if can use reverse product rule on LHS

$$-\frac{dx}{dt} + \frac{4x}{200+t} \qquad \frac{d}{dt}(x) = \frac{dx}{dt} \checkmark \qquad \frac{d}{dt}(1) \neq \frac{4}{200+t} \times$$

$\therefore$ can't use reverse product rule

-Only option is to introduce I.F : $e^{\int P\,dt} = e^{\int \frac{4}{200+t}dt}$

$= e^{4\ln(200+t)} = e^{\ln(200+t)^4}$

$= (200+t)^4$

multiplying by I.F

$$(200+t)^4\frac{dx}{dt} + (200+t)^{\cancel{4}\,3}\left(\frac{4x}{\cancel{200+t}}\right) = 50(200+t)^4$$

$\frac{d}{dt}(x) = \frac{dx}{dt}(v)$ $\qquad \frac{d}{dt}(200+t)^4 = 4(200+t)^3(v)$

and checking for reverse product rule ($\checkmark$)

$\therefore$ rewrite LHS as

$$\frac{d}{dt}\left(x(200+t)^4\right) = 50(200+t)^4$$

integrate both sides

$$x(200+t)^4 = \int 50(200+t)^4\,dt$$

$\boxed{G.S}$ $x(200+t)^4 = 10(200+t)^5 + c$

subbing in logical initial conditions:

when $t=0, x=0$

$$0(200+0)^4 = 10(200+0)^5 + c$$

$\Rightarrow 1.6\times10^9 = 10(3.2\times10^{11}) + c$

$\Rightarrow 1.6\times10^9 = 3.2\times10^{12} + c$

$\Rightarrow c = -3.199\ldots\times10^{12}$

$= -3.2\times10^{12}$

$\boxed{P.S}$ $x(200+t)^4 = 10(200+t)^5 - 3.2\times10^{12}$

and subbing in $t=8$

$$x(200+8)^4 = 10(200+8)^5 - 3.2\times10^{12}$$

$\div(200+8)^4$

$$x = 10(200+8) - \frac{3.2\times10^{12}}{(200+8)^4}$$

$= 370.3916\ldots = \boxed{370g \ (3\text{ s.f})}$

Question 5 continued

(c). unrealistic that the pollutant dissolves straight away upon entry

· the rate of leaking could be made to vary with the volume of water in the pond

(Total for Question 5 is 10 marks)

6.

$$f(x) = \frac{x+2}{x^2+9}$$

(a) Show that

$$\int f(x)\,dx = A\ln(x^2+9) + B\arctan\left(\frac{x}{3}\right) + c$$

where c is an arbitrary constant and A and B are constants to be found.

(4)

(b) Hence show that the mean value of $f(x)$ over the interval $[0, 3]$ is

$$\frac{1}{6}\ln 2 + \frac{1}{18}\pi$$

(3)

(c) Use the answer to part (b) to find the mean value, over the interval $[0, 3]$, of

$$f(x) + \ln k$$

where k is a positive constant, giving your answer in the form $p + \frac{1}{6}\ln q$, where p and q are constants and q is in terms of k.

(2)

(a) notice asked to integrate a fractional expression - looking at the three ways to do this:

Fractional expressions

4a. Can I split the numerator?
Is there a single term in the denominator?
4b. Can I do partial fractions?
Does the denominator factorise?
4c. Can I do algebraic division?
Is the fraction improper?

explained more in detail on pg. 21 (end of question)

...can split the numerator and evaluate the two separate integrals:

$$\int \frac{x+2}{x^2+9}\,dx = \int \frac{x}{x^2+9}\,dx \;①\; + 2\int \frac{1}{x^2+9}\,dx \;②$$

① using integration by reverse chain rule (because the numerator is a scalar multiple of the derivative of denominator)

consider: $\ln(x^2+9)$

differentiate (CHAIN RULE): $\frac{2x}{x^2+9}$

scale: $\times \frac{1}{2}$

$\therefore \frac{1}{2}\ln(x^2+9)$

② looking at the different formula book integrations

$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\ln\cosh x$
$\dfrac{1}{\sqrt{a^2-x^2}}$	$\arcsin\left(\dfrac{x}{a}\right) \quad (\lvert x\rvert < a)$
$\dfrac{1}{a^2+x^2}$	$\dfrac{1}{a}\arctan\left(\dfrac{x}{a}\right)$
$\dfrac{1}{\sqrt{x^2-a^2}}$	$\operatorname{arcosh}\left(\dfrac{x}{a}\right),\ \ln\{x+\sqrt{x^2-a^2}\} \quad (x > a)$
$\dfrac{1}{\sqrt{a^2+x^2}}$	$\operatorname{arsinh}\left(\dfrac{x}{a}\right),\ \ln\{x+\sqrt{x^2+a^2}\}$
$\dfrac{1}{a^2-x^2}$	$\dfrac{1}{2a}\ln\left\lvert\dfrac{a+x}{a-x}\right\rvert = \dfrac{1}{a}\operatorname{artanh}\left(\dfrac{x}{a}\right) \quad (\lvert x\rvert < a)$
$\dfrac{1}{x^2-a^2}$	$\dfrac{1}{2a}\ln\left\lvert\dfrac{x-a}{x+a}\right\rvert$

$a^2 = 9$

$a = 3$

$\Rightarrow \frac{2}{3}\arctan\left(\frac{x}{3}\right)$

$$\boxed{\Rightarrow \int \frac{x+2}{x^2+9}\,dx = \frac{1}{2}\ln(x^2+9) + \frac{2}{3}\arctan\left(\frac{x}{3}\right) + c}$$

(b) remembering the formula for mean value of a function:

$\overline{f(x)} = \frac{1}{b-a}\int_a^b f(x)\,dx$ - subbing in limits in question

$$= \frac{1}{3-0}\int_0^3 \frac{x+2}{x^2+9}\,dx = \frac{1}{3}\int_0^3 \frac{x+2}{x^2+9}\,dx$$

know the indefinite integration of above from part (a) - need to evaluate this at the limits

$$= \frac{1}{3}\left[\frac{1}{2}\ln(x^2+9) + \frac{2}{3}\arctan\left(\frac{x}{3}\right)\right]_0^3$$

$$= \frac{1}{3}\left\{\left[\frac{1}{2}\ln((3)^2+9) + \frac{2}{3}\tan^{-1}\left(\frac{3}{3}\right)\right] - \left[\frac{1}{2}\ln(0^2+9) + \frac{2}{3}\tan^{-1}(0)\right]\right\}$$

$$= \frac{1}{3}\left[\frac{1}{2}\ln(18) + \frac{2}{3}\left(\frac{\pi}{4}\right)\right] - \left[\frac{1}{2}\ln(9) + 0\right]\}$$

$$= \frac{1}{3}\left(\frac{1}{2}\ln(18) - \frac{1}{2}\ln(9) + \frac{\pi}{6}\right)$$

using quotient rule on the logs

$$= \frac{1}{3}\left(\frac{1}{2}\ln\left(\frac{18}{9}\right) + \frac{\pi}{6}\right)$$

$$\boxed{= \frac{1}{6}\ln(2) + \frac{\pi}{18}}$$

(c) notice now we're asked to evaluate a geometric consideration - ∴ have to increase the mean value by $\ln(k)$

$= \frac{1}{6}\ln(2) + \frac{\pi}{18} + \ln(k)$ but need all ln terms together, so get common denominator

Question 6 continued

$$= \frac{1}{6}\ln(2) + \frac{6}{6}\ln(k) + \frac{\pi}{18}$$

factorise $\frac{1}{6}$ out

$$= \frac{1}{6}\left(\ln(2) + 6\ln(k)\right) + \frac{\pi}{18}$$

using log power and addition rule

$$= \frac{1}{6}\left(\ln 2k^6\right) + \frac{\pi}{18}$$

$$\boxed{\Rightarrow p = \pi/18,\ q = 2k^6}$$

Reminders:

Students find fractions tough as fractions can be so many types.

Check first (and throughout the question) if you can simplify by:

- using basic indices rules to simplify and expand brackets
  - $x^a \times x^b = x^{a+b}$
  - $\frac{x^a}{x^b} = x^{a-b}$
  - $\frac{3}{5x}$ *means* $\frac{3}{5}x^{-1}$.
  - $(\sqrt[b]{x})^a$ or $\sqrt[b]{x^a} = x^{\frac{a}{b}}$
- Factorising and maybe cancel first
- Is there a single term in denominator?

  split fractions using $\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$ or $(a+b)c^{-1}$

Then ask yourself:

1. **Is it an easy power type?** $\int x^n dx = \frac{x^{n+1}}{n+1}$
2. **Is it ln (natural logarithm)?** Form $\int \frac{f'(x)}{f(x)}dx$

   To recognize these, the power in the denominator is (almost always) 1. When you bring the denominator up to the numerator using negative power indices rule you get a power of -1. By adding one to the power and dividing it, you'll end up dividing by zero which you can't do

   $$\int \frac{f'(x)}{f(x)}dx = \ln f(x) + C$$

   Method: copy ln(denominator). Remember ignore then differentiate to check you get what is inside the integral - correct with numbers only, not variables and only correct by multiplying or dividing. We can ignore the pink part since the derivative 'pops' out when we differentiate and we know when we differentiate our answer it must be what is inside the integral).
3. **Is it bring up and harder power type?** Bring the power up and becomes the form $\int f'(x)f(x)^n\,dx = \frac{f(x)^{n+1}}{n+1} + C$

   Recognisable by a power in the denominator other than

   $$\int \frac{4x}{(2x^2-1)^3} = \int 4x(2x^2-10)^{-3}\,dx \text{ etc}$$
4. **Is it Partial fractions!** Recognisable by products in the denominator.

   Form 1 $\frac{\dots}{(cx+d)(ex+f)} = \frac{A}{cx+d} + \frac{B}{ex+f}$

   Form 2 $\frac{\dots}{(dx+e)(fx+g)^2} = \frac{A}{dx+e} + \frac{B}{fx+g} + \frac{C}{(fx+g)^2}$

   (only advanced courses have this form)

   Form 3 $\frac{\dots}{(dx+e)(fx^2+g)} = \frac{A}{dx+e} + \frac{Bx+C}{fx^2+g}$
5. **Is it divide first?** Recognisable by **two or more terms** in the denominator and also where we have the matching highest powers in both numerator and denominator or a higher power in the numerator
6. **Rewriting/adapting fraction in a clever way (split up the numerator to get two fractions)**
7. **Is it inverse trig? (may need to complete the square first)** Either use the inverse trig results below or use a trig substitution

$$\int \frac{1}{\sqrt{a^2-(bx)^2}}\,dx = \frac{1}{b}\sin^{-1}\left(\frac{bx}{a}\right) + C$$

$$\int -\frac{1}{\sqrt{a^2-(bx)^2}}\,dx = \frac{1}{b}\cos^{-1}\left(\frac{bx}{a}\right) + C$$

$$\int \frac{1}{a^2+(bx)^2}\,dx = \frac{1}{ab}\tan^{-1}\left(\frac{bx}{a}\right) + C$$

(Total for Question 6 is 9 marks)

7.

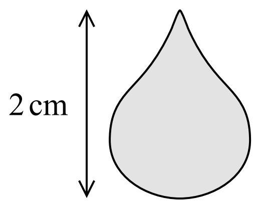


Figure 2

Figure 2 shows the image of a gold pendant which has $\boxed{\text{height } 2\,\text{cm}}$. The pendant is modelled by a solid of revolution of a curve C about the y-axis. The curve C has parametric equations

$$\boxed{x = \cos\theta + \frac{1}{2}\sin 2\theta,} \quad \boxed{y = -(1 + \sin\theta)} \qquad 0 \leqslant \theta \leqslant 2\pi$$

(a) Show that a Cartesian equation of the curve C is

$$x^2 = -(y^4 + 2y^3)$$

(4)

(b) Hence, using the model, find, in cm³, the volume of the pendant.

(4)

(a) this should remind us of converting trig parametric equations to Cartesian equations from Pure Yr 2 :

want to manipulate each of

$x = f(\theta)$ and $y = g(\theta)$ to then sub into KNOWN IDENTITY:

$$\sin^2\theta + \cos^2\theta \equiv 1$$

... x function first:

manipulate using sin double angle - $\sin 2\theta = 2\sin\theta\cos\theta$

$$x = \cos\theta + \frac{1}{2}(2\sin\theta\cos\theta)$$

$$x = \cos\theta + \sin\theta\cos\theta$$

factorise $\cos\theta$ out

$$x = \cos\theta(\underbrace{1 + \sin\theta})$$

notice this is '$-y$'

$$x = \cos\theta(-y)$$

$\div -y \qquad \div -y$

$$\Rightarrow \boxed{\cos\theta = -\frac{x}{y}}$$

... y function next:

$$y = -1 - \sin\theta \quad \Rightarrow \boxed{\sin\theta = -y - 1}$$

Question 7 continued

Subbing into trig identity

$$\left(\frac{-x}{y}\right)^2 + (-y-1)^2 \equiv 1$$

expand

$$\frac{x^2}{y^2} + y^2 - 2y + 1 = 1$$

$$\Rightarrow \frac{x^2}{y^2} = -y^2 + 2y$$

$\times y^2$ $\times y^2$

$$\Rightarrow x^2 = -y^4 + 2y^3$$

factorise -1 out

$$\boxed{\Rightarrow x^2 = -(y^4 - 2y^3)}$$

(b) notice need to sub into formula for volumes of revolution about the 'y-axis' : $V = \pi \int_\alpha^\beta x^2 \, dy$

$$\Rightarrow V = \pi \int_{-2}^{0} -(y^4 + 2y^3) \, dy$$

integrate above

$$= \pi \left[-\left(\frac{y^5}{5}\right) + \frac{1}{2}y^4 \right]_{-2}^{0}$$

$$\Rightarrow -\pi \left\{ \left[\frac{0^5}{5} + \frac{1}{2}(0)^4\right] - \left[\left(\frac{(-2)^5}{5} + \frac{1}{2}(-2)^4\right)\right] \right\}$$

$$= -\pi\left(-\frac{8}{5}\right) = \boxed{\frac{8\pi}{5}\text{ cm}^3}$$

NOTE: usually could've used the volumes of revolution formula for parametrically defined curves - but the fact that we're asked to find the Cartesian equation for C hints we should use the standard formula for volumes of revolution about the y-axis AND using the vol. of rev. for parametrics gets really horrible

Question 7 continued

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Question 7 continued

(Total for Question 7 is 8 marks)

8. The line l_1 has equation $\dfrac{x-2}{4} = \dfrac{y-4}{-2} = \dfrac{z+6}{1}$

The plane Π has equation $x - 2y + z = 6$

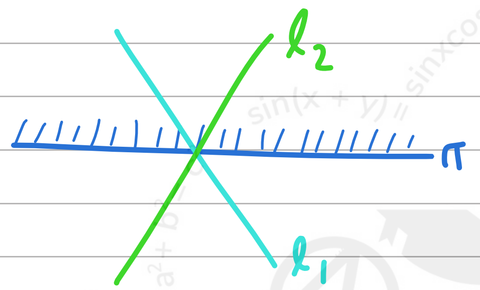
The line l_2 is the reflection of the line l_1 in the plane Π.

Find a vector equation of the line l_2

(7)

notice we're asked to reflect the line l_1 in the plane Π

↳ aim to get 2 points that are on the reflected l_1 (i.e l_2) and find the vector equation through them



↳ first point: point of intersection between l_1 and Π → notice given Π in Cartesian form. need its scalar product form

$$\Pi : \mathbf{r} \cdot \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = 6$$

and l_1 in Cartesian – need its vector parametric form (negated numerator = position vector, denominator = direction vector)

$$l_1 : \mathbf{r} = \begin{pmatrix} 2 \\ 4 \\ -6 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix}$$

of which general coordinate:

$$\mathbf{r} = \begin{pmatrix} 2+4\lambda \\ 4-2\lambda \\ -6+\lambda \end{pmatrix}$$

for p.o.i – sub l_1 into Π

$$\begin{pmatrix} 2+4\lambda \\ 4-2\lambda \\ -6+\lambda \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = 6$$

evaluate dot product:

$$2+4\lambda-2(4-2\lambda)+(-6+\lambda)=6$$

expand

$$2+4\lambda-8+4\lambda-6+\lambda=6$$

collect like terms:

$$9\lambda = 18$$

$\div 9 \qquad \div 9$

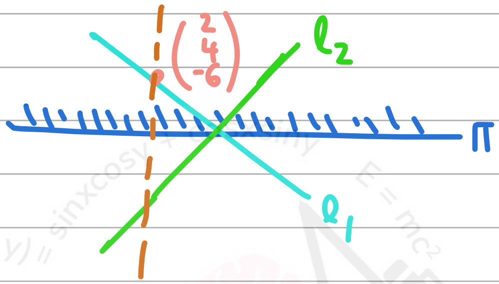
$$\Rightarrow \boxed{\lambda = 2}$$

Question 8 continued

sub into general ℓ_1 coordinate to get p.o.i

$$\begin{pmatrix} 2+4(2) \\ 4-2(2) \\ -6+2 \end{pmatrix} = \boxed{\begin{pmatrix} 10 \\ 0 \\ -4 \end{pmatrix}} \text{ - call it A}$$

second point : reflection of a given point (position vector of ℓ_1) through the plane



can do this by finding the p.o.i between the plane and the line perpendicular to the plane - its position vector $= \begin{pmatrix} 2 \\ 4 \\ -6 \end{pmatrix}$

direction vector (NORMAL to plane - from its cartesian equation) $= \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

$$\Rightarrow \ell_{perp.} = \begin{pmatrix} 2 \\ 4 \\ -6 \end{pmatrix} + t\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

general coordinate: $\begin{pmatrix} 2+t \\ 4-2t \\ -6+t \end{pmatrix}$

need p.o.i between $\ell_{perp.}$ and Π - sub general coordinate into scalar product for Π

$$\begin{pmatrix} 2+t \\ 4-2t \\ -6+t \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = 6$$

evaluate dot product

$$2+t-2(4-2t)+(-6+t)=6$$

expand brackets

$$2+t-8+4t-6+t=6$$

collect like terms

$$6t = 18$$

$\div 6 \quad \div 6$

$$\Rightarrow \boxed{t=3}$$

considering if from point to plane $t=3$, then from plane to reflected

Question 8 continued

point this must be another $t=3$ $\therefore$ the point on l_{perp} where $t=6$

subbing this in

$$l_{perp}: \begin{pmatrix} 2+6 \\ 4-2(6) \\ -6+6 \end{pmatrix} = \boxed{\begin{pmatrix} 8 \\ -8 \\ 0 \end{pmatrix}} \text{ - call it } B$$

now just need vector equation through these:

take ANY of the two points, A or B as the position vector and the direction

$$\text{vector} = \begin{pmatrix} 10 \\ 0 \\ -4 \end{pmatrix} - \begin{pmatrix} 8 \\ -8 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 8 \\ -4 \end{pmatrix} \div 2 = \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix}$$

$$\boxed{\Rightarrow r = \begin{pmatrix} 10 \\ 0 \\ -4 \end{pmatrix} + k\begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix}}$$

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Question 8 continued

(Total for Question 8 is 7 marks)

9. A company plans to build a new fairground ride. The ride will consist of a capsule that will hold the passengers and the capsule will be attached to a tall tower. The capsule is to be released from rest from a point half way up the tower and then made to oscillate in a vertical line.

The vertical displacement, x metres, of the top of the capsule below its initial position at time t seconds is modelled by the differential equation,

$$m\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + x = 200\cos t, \qquad t \geqslant 0$$

where m is the mass of the capsule including its passengers, in thousands of kilograms.

The maximum permissible weight for the capsule, including its passengers, is 30 000 N.

Taking the value of g to be 10 ms^{-2} and assuming the capsule is at its maximum permissible weight,

(a) (i) explain why the value of m is 3

(ii) show that a particular solution to the differential equation is

$$x = 40\sin t - 20\cos t$$

(iii) hence find the general solution of the differential equation.

(8)

(b) Using the model, find, to the nearest metre, the vertical distance of the top of the capsule from its initial position, 9 seconds after it is released.

(4)

(a)(i) see that mass, weight and 'g' are all LINKED by the formula: $W = mg$

subbing in $W_{max} = 30{,}000$ - but in thousands so $\boxed{30\text{kg}}$

and $\boxed{g = 10}$

$30 = m(10)$

$\div 10 \quad \div 10$

$\Rightarrow \boxed{m = 3}$

$\Rightarrow 3\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + x = 200\cos t$

(ii) WAY 1: by substitution into 2ODE

noticing that we could sub the given P.S into LHS of the 2ODE - if it equals the RHS then we've proved it's a P.S

$x = 40\sin t - 20\cos t$

$\frac{dx}{dt} = 40\cos t + 20\sin t$

$\frac{d^2x}{dt^2} = -40\sin t + 20\cos t$

Question 9 continued

LHS :

$$3(-40\sin t + 20\cos t) + 4(40\cos t + 20\sin t) + (40\sin t - 20\cos t)$$

expand brackets

$$-\cancel{120}\sin t + 60\cos t + 160\cos t + \cancel{80}\sin t + \cancel{40}\sin t - 20\cos t$$

collect like terms

$$= 200\cos t = RHS \quad \therefore x = 40\sin t - 20\cos t$$

WAY 2: using P.I table

Form of f(x)	Form of particular integral
k	λ
$ax + b$	$\lambda + \mu x$
$ax^2 + bx + c$	$\lambda + \mu x + \nu x^2$
ke^{px}	λe^{px}
$m\cos \omega x$	$\lambda \cos \omega x + \mu \sin \omega x$
$m \sin \omega x$	$\lambda \cos \omega x + \mu \sin \omega x$
$m \cos \omega x + n \sin \omega x$	$\lambda \cos \omega x + \mu \sin \omega x$

WARNING!
The particular integral must not contain any term in the complementary function. If it does, you'll need to add an x and possibly even an x^2 in front of your usual PI form

$$\boxed{\text{let } x = \lambda\cos t + \mu\sin t}$$

$$\frac{dx}{dt} = -\lambda\sin t + \mu\cos t$$

$$\frac{d^2x}{dt^2} = -\lambda\cos t - \mu\sin t$$

sub into 2ODE

$$3(-\lambda\cos t - \mu\sin t) + 4(-\lambda\sin t + \mu\cos t) + \lambda\cos t + \mu\sin t = 200\cos t$$

expand brackets

$$-3\lambda\cos t - 3\mu\sin t - 4\lambda\sin t + 4\mu\cos t + \lambda\cos t + \mu\sin t = 200\cos t$$

...comparing sines:

$$-3\mu - 4\lambda + \mu = 0$$

collect like terms

$$-2\mu - 4\lambda = 0 \quad ①$$

...comparing cos:

$$-3\lambda + 4\mu + \lambda = 200$$

collect like terms

$$-2\lambda + 4\mu = 200 \quad ②$$

solve ① and ② simultaneously

②×2 − ①

$$8\mu - 4\lambda = 400$$
$$-\quad -2\mu - 4\lambda = 0$$
$$10\mu = 400$$
$$\div 10 \qquad \div 10$$
$$\boxed{\mu = 40}$$

sub into ① $\quad -2(40) - 4\lambda = 0$

$$\Rightarrow 4\lambda = -80$$
$$\div 4 \qquad \div 4$$

Question 9 continued

$\boxed{\lambda = -20}$

Subbing into initial 'x' P.I

$\Rightarrow$ $\boxed{\text{P.I}}$ $\boxed{x = 40\sin t - 20\cos t}$

(iii) $\boxed{\text{A.E}}$ $3m^2 + 4m + 1 = 0$

solve - calc equation solver or quadratic formula

$$m = \frac{-4 \pm \sqrt{(4)^2 - 4(3)(1)}}{2(3)}$$

$$= \frac{-4 \pm \sqrt{16-12}}{6} = \frac{-4 \pm \sqrt{4}}{6}$$

$$= \frac{-4 \pm 2}{6}$$

$\Rightarrow m = -1/3$ or -1

↳ two real roots - so subbing into associated general A.E formula:

$x = Ae^{\alpha t} + Be^{\beta t}$; $x = Ae^{-t} + Be^{-1/3 t}$

G.S = A.E + P.I

↳ from (a)(ii)

$\boxed{\boxed{\text{G.S}} : x = Ae^{-t} + Be^{-1/3 t} + 40\sin t - 20\cos t}$

(b) 'using model' suggests have to sub initial conditions in:

at $t = 0$, $x = 0$

$0 = Ae^0 + Be^0 + 40\sin(0) - 20\cos(0)$

$\Rightarrow$ $0 = A + B - 20$

$\Rightarrow$ $A + B = 20$ - ①

and at $t = 0$, $\frac{dx}{dt} = 0$:

differentiate G.S - using $\frac{d}{dt}(e^{kt}) = ke^{kt}$

$$\frac{dx}{dt} = -Ae^{-t} - \frac{1}{3}Be^{-1/3 t} + 40\cos t + 20\sin t$$

$\Rightarrow$ $0 = -A - 1/3 B + 40$

$\Rightarrow$ $A + 1/3 B = 40$ - ②

Question 9 continued

solve simultaneously - calc equtn solver or by substitution

① - ②

$$\begin{aligned} & A + B = 20 \\ - \; & A + \tfrac{1}{3}B = 40 \\ \hline & \tfrac{2}{3}B = -20 \end{aligned}$$

$\div \tfrac{2}{3} \qquad \div \tfrac{2}{3}$

$\Rightarrow \boxed{B = -30}$ Sub into ①

$20 = A - 30$

$\Rightarrow \boxed{A = 50}$

subbing these into G.S (part (iii))

$x = 50e^{-t} - 30e^{-\frac{1}{3}t} + 40\sin t - 20\cos t$

subbing when $t = 9$

$x = 50e^{-9} - 30e^{-\frac{1}{3}(9)} + 40\sin(9) - 20\cos(9)$

$= \boxed{33\text{m}}$

(Total for Question 9 is 12 marks)

TOTAL FOR PAPER IS 75 MARKS

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